FUNDAMENTAL SOLUTIONS OF THE DYNAMICAL EQUATIONS OF ELASTICITY FOR NONHOMOGENEOUS MEDIA

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The fundamental tensor for the dynamical equations of the theory of elasticity for nonhomogeneous isotropic media is constructed. This problem was posed in [1]. The construction is carried out along the lines of the developments in [2]. The immediate application of the results of [2] is not possible here, because the three-dimensional equations of elasticity possess multiple characteristics.

In problems in the theory of elastic vibrations an important role is played by the so-called point sources of vibrations: concentrated forces in infinite space, centers of expansion, double forces, concentrated couples, concentrated moments, and so forth.

The known fundamental solution of Volterra for the equations of elasticity represents, as may be easily shown, a combination of a center of expansion and concentrated moments, with corresponding moment axes along the coordinate axes.

A knowledge of the displacements corresponding to an arbitrary concentrated force enables one to determine easily the displacements corresponding to an arbitrary point-source.

For a homogeneous elastic medium, the problem of the determination of the effect of a concentrated force varying in an arbitrary manner but always directed along the x-axis has been solved in finite form in [3]. The corresponding problem for a nonhomogeneous medium is considered below.

1. Formulation of the problem. Suppose that a concentrated force acts at a point M_0, its magnitude being given by the function $\chi(t)$.

It suffices to consider the special case of a concentrated impulse when $\chi(t) = \delta(t)$, where $\delta(t)$ is Dirac's delta function, since the displacement vector $\mathbf{u}_j$ in the general case of the function $\chi(t)$ is given, in terms of the displacement vector $\mathbf{h}_j$ corresponding to a concentrated impulse, by the formula

$$\mathbf{u}(M, t) = \int_0^t \mathbf{h}_j(M, t-t')\,\chi(t')\,dt'$$

The components of the vectors $\mathbf{h}_j(M, t)$ constitute the elements of a tensor $H(M, t) = \| h_{ij}(M, t) \|$, which is called the fundamental tensor of the theory of elasticity.

Let us formulate the mathematical problem of the determination of the vectors $\mathbf{h}_j$. Let $u_i(x_1, x_2, x_3, t)$ be the components of the displacement vector $\lambda = \lambda(x_1, x_2, x_3)$ and $\mu = \mu(x_1, x_2, x_3)$ be Lamé's parameters, $\rho = \rho(x_1, x_2, x_3)$ be the density of the medium. It will be assumed that λ, μ, ρ are analytic functions of x_1, x_2, x_3.

The equations of the theory of elasticity may then be written (see [1]):

$$\mathbf{Lu} = \rho\mathbf{u}_{tt} - (\lambda+\mu)\,\mathrm{grad\,div}\,\mathbf{u} - \mu\Delta\mathbf{u} - \mathrm{div}\,\mathbf{u}\,\mathrm{grad}\,\lambda - 2D\,\mathrm{grad}\,\mu = \mathbf{K} \quad (1.1)$$

where

$$D = \|\varepsilon_{ij}\| = \left\| \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\|$$

is the strain tensor.

Let us set $\mathbf{u} = 0$ for $t < 0$ and consider a sequence of body force vectors $\mathbf{K}_\epsilon$ such that

$$\mathbf{K}_\varepsilon = 0, \quad r = |MM_0| > \varepsilon, \quad M = M(x_1, x_2, x_3), \quad M_0 = M_0(x_1^0, x_2^0, x_3^0)$$

$$\int \mathbf{K}_\varepsilon\, dx_1\, dx_2\, dx_3 = \chi(t)\,\mathbf{i}_j$$

Then, in the limit as $\epsilon \to 0$, the corresponding displacement vector will correspond to a concentrated force directed along the x_j-axis and having magnitude $\chi(t)$. The limiting vector of these body forces is

$$\mathbf{K} = \delta(M - M_0)\,\chi(t)\,\mathbf{i}_j \quad (1.2)$$

where $\delta(M - M_0)$ is a δ-function with singularity at the point M_0. Putting $\chi(t) = \delta(t)$ in (1.2) we obtain the body-force vector which corresponds to the vector $\mathbf{h}_j$.*

The vector $\mathbf{h}_j$ may be sought as a solution of the Cauchy problem

$$\mathbf{L}\mathbf{h}_j = 0; \qquad \mathbf{h}_j|_{t=0} = 0, \quad \frac{\partial \mathbf{h}_j}{\partial t}|_{t=0} = \frac{1}{\rho(M_0)}\delta(M - M_0)\,\mathbf{i}_j \tag{1.3}$$

since this vector, when defined to be 0 for $t < 0$, yields, when operated upon by the operator L, the body-force vector (1.2), with $\chi(t) = \delta(t)$.

By means of the fundamental solution tensor the integration of the Cauchy problem for the equations of elasticity is reduced to quadratures.

Indeed, for the equations of the theory of elasticity one has the analog of the formula of Green for harmonic functions, the Green-Volterra formula

$$\int_T (\mathbf{v}L(\mathbf{u}) - \mathbf{u}\mathbf{L}(\mathbf{v}))\,dx_1dx_2dx_3dt = \int_S (\mathbf{v}\mathbf{p}(\mathbf{u}) - \mathbf{u}\mathbf{p}(\mathbf{v}))\,dS$$
$$(\mathbf{p}\mathbf{u} = \rho\mathbf{u}_t - \sigma_{kj}n_j\mathbf{i}_k) \tag{1.4}$$

where σ_{kj} are the components of the stress tensor, n_j are the components of the four-dimensional normal, S is the hypersurface which bounds the four-dimensional volume T and here and in the following repeated indices indicate that a summation is to be performed.

Let T denote the half-space $t \geqslant 0$, and

$$\mathbf{v} = \mathbf{h}_j(M - M_0,\ t_0 - t), \qquad M = M(x_1, x_2, x_3), \qquad M_0 = M_0(x_1^\circ, x_2^\circ, x_3^\circ)$$

and $\mathbf{u}$ be the solution of the Cauchy problem

$$\mathbf{L}\mathbf{u} = \mathbf{K}; \qquad \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t=0} = \mathbf{u}_1 \tag{1.5}$$

Since

$$\mathbf{L}\mathbf{v} = \delta(M - M_0)\,\delta(t_0 - t)$$

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* Other point-sources correspond to other "delta-like" body forces. For example, a center of expansion corresponds to

$$\operatorname{grad}\delta(M - M_0)\,\chi(t)$$

while a double force corresponds to

$$\chi(t)\frac{\partial}{\partial x_j}\delta(M - M_0)\,\mathbf{i}_j \text{ etc.}$$

the integration of the expression $\mathbf{u}\mathbf{L}\mathbf{h}_j$ yields the jth component of the vector $\mathbf{u}$, and we have

$$u_j(M_0, t_0) = \frac{\partial}{\partial t_0}\int h_{ij} u_{0i}\, dM + \int h_{ij} n_{1i}\, dM + \int\limits_{0\leq t\leq t_0} h_{ij} K_j\, dM\, dt \qquad (dM = dx_1 dx_2 dx_3)$$

or, what is the same

$$\mathbf{u} = \frac{\partial}{\partial t_0}\int H(M - M_0, t_0)\,\mathbf{u}_0(M)\, dM + \int H(M - M_0, t_0)\,\mathbf{u}_1(M)\, dM + \int\limits_{0\leq t\leq t_0} H(M - M_0, t_0 - t)\,\mathbf{K}(M_1 t)\, dM\, dt \tag{1.6}$$

where $H = \| h_{ij} \|$ is the fundamental tensor.

In order to construct the fundamental tensor it is only necessary to solve the Cauchy problem (1.3). Let us now express the δ_j-function in terms of plane waves (see [2,5,6])

$$\delta(x_1 - x_1^\circ, x_2 - x_2^\circ, x_3 - x_3^\circ) = -\frac{1}{8\pi^2}\iint \delta''(\omega_l(x_l - x_l^\circ))\, d\omega$$
$$(\omega_1^2 + \omega_2^2 + \omega_3^2 = 1)$$

and suppose that $\mathbf{h}_{\omega j}$ is the solution of the Cauchy problem

$$\mathbf{L}(\mathbf{h}_{\omega j}) = 0 \tag{1.7}$$

$$\mathbf{h}_{\omega j}|_{t=0} = 0, \quad \left.\frac{\partial \mathbf{h}_{\omega j}}{\partial t}\right|_{t=0} = -\frac{1}{8\pi^2\rho(M_0)}\delta''(\omega_l(x_l - x_l^\circ))\,\mathbf{i}_j \tag{1.8}$$

Then, obviously, the sought vector $\mathbf{h}_j$ is given by the formula

$$\mathbf{h}_j(M, t) = \int\limits_{|\omega|=1} \mathbf{h}_{\omega j}\, d\omega \tag{1.9}$$

The solution of the Cauchy problem (1.7), (1.8) will be constructed by means of "ray" solutions.

2. "Ray" solutions of the dynamical equations of elastic body (see [6,7]). Let f_0 be an arbitrary function and f_k be a sequence of its iterated integrals

$$f_k(x) = \int f_{k-1}(x)\, dx \tag{2.1}$$

We shall seek solutions of the equation $\mathbf{L}\mathbf{u} = 0$ of the form

$$\mathbf{u} = \sum_{k=0}^{\infty} \mathbf{u}_k(x_1, x_1, x_3, t)\, f_k(\gamma(x_1, x_2, x_3, t)), \tag{2.2}$$

where γ is a certain fixed function. Setting $\mathbf{K} = 0$ in (1.1), substituting for $\mathbf{u}$ its expression from (2.2), and then equating the coefficient of each function f_k to zero, we obtain

$$\mathbf{N}\mathbf{u}_{k+2} + \mathbf{M}\mathbf{u}_{k+1} + \mathbf{L}\mathbf{u}_k = 0 \qquad (\mathbf{u}_{-1} = \mathbf{u}_{-2} \equiv 0) \qquad (k = -2, -1, 0, 1, 2, \ldots) \tag{2.3}$$

where the operator $\mathbf{L}$ is the same as in Equation (1.1) and the operators N and M are defined as follows:

$$\mathbf{N}\mathbf{u} = (\rho\gamma_t^2 - \mu(\operatorname{grad}\gamma)^2)\mathbf{u} - (\lambda + \mu)\operatorname{grad}\gamma\,(\operatorname{grad}\gamma\mathbf{u}) \tag{2.4}$$

$$\begin{aligned}\mathbf{M}\mathbf{u} = {}& 2\rho\mathbf{u}_t\gamma_t - (\lambda + \mu)\,[\operatorname{div}\mathbf{u}\operatorname{grad}\gamma + \operatorname{grad}(\mathbf{u}\operatorname{grad}\gamma)] - \\ & - \mu\,[\mathbf{u}\Delta\gamma + 2(\operatorname{grad}\mathbf{u}_k\operatorname{grad}\gamma)\,i_k] - \operatorname{grad}\lambda\,(\mathbf{u}\operatorname{grad}\gamma) - \\ & - (\operatorname{grad}\mu\mathbf{u})\operatorname{grad}\gamma - (\operatorname{grad}\mu\operatorname{grad}\gamma)\,\mathbf{u}\end{aligned} \tag{2.5}$$

$$\operatorname{grad}\gamma = (\gamma_{x_1},\ \gamma_{x_2},\ \gamma_{x_3}) \tag{2.6}$$

Setting $k = -2$ in (2.3) we obtain $\mathbf{N}\mathbf{u}_0 = 0$. It is natural to assume that $\mathbf{u}_0 \neq 0$, hence the determinant of the coefficients of this linear system of algebraic equations must be zero. From this it follows that we must have one of the following relations: either (longitudinal wave case)

$$(\operatorname{grad}\gamma)^2 = \frac{1}{a^2}\gamma_t^2, \qquad a = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad \mathbf{u}_0 \parallel \operatorname{grad}\gamma \tag{2.7}$$

or (transverse wave case)

$$(\operatorname{grad}\gamma)^2 = \frac{1}{b^2}\gamma_t^2, \qquad b = \sqrt{\frac{\mu}{\rho}}, \qquad \mathbf{u}_0 \perp \operatorname{grad}\gamma \tag{2.8}$$

An important role in the study of the equation

$$(\operatorname{grad}\gamma)^2 = \frac{1}{c^2(x, y, z)}\gamma_t^2 \tag{2.9}$$

is played by the extremals of Fermat's functional

$$\tau = \int_{M_1}^{M} \frac{ds}{c} \tag{2.10}$$

If M_1 is a fixed point and the integral (2.9) is taken along extremals, then the quantity τ may be used to characterize points on the extremals. A curve in four-dimensional space $(x_1,\ x_2,\ x_3,\ t)$

$$x_1 = x_1(\tau), \qquad x_2 = x_2(\tau), \qquad x_3 = x_3(\tau), \qquad t = \tau + \text{const} \tag{2.11}$$

will be a characteristic of Equation (2.9).

Let us pass through each point M_1 of a fixed surface Σ a perpendicular extremal of the integral (2.10); a point of each such extremal is characterized by the corresponding value of the parameter τ, while the points M_1 are characterized by the two surface parameters α and β. Thus, in a neighborhood of the surface Σ which constitutes a regular field of extremals we may introduce curvilinear coordinates α, β, γ such that

$$x_i = x_i(\alpha, \beta, \tau) \quad (i = 1, 2, 3) \qquad \text{or} \qquad \mathbf{x} = \mathbf{x}(\alpha, \beta, \tau) \tag{2.12}$$

Consider the longitudinal wave case. From Formulas (2.3) to (2.7) it follows that

$$\mathbf{u}_2 = 0, \qquad \mathbf{u}_{-1} = 0, \qquad \mathbf{u}_0^\circ = 0$$

where $\mathbf{u}_0^\circ$ is the component of the vector $\mathbf{u}_0$ which is perpendicular to the ray.

Now suppose that the vectors $\mathbf{u}_{-2}$, $\mathbf{u}_{-1}$, ..., $\mathbf{u}_k$, $\mathbf{u}_{k+1}^\circ$ are known (where $\mathbf{u}_{k+1}^\circ$ is the component of the vector $\mathbf{u}_{k+1}$ which is perpendicular to the ray). If we make use of (2.3) and

$$\begin{gathered} \mathbf{u}_{k+1} = \overset{\circ}{\mathbf{u}}_{k+1} + \varphi_{k+1} \operatorname{grad} \gamma, \qquad \mathbf{u}_{k+2} = \overset{\circ}{\mathbf{u}}_{k+2} + \varphi_{k+2} \operatorname{grad} \gamma \\ \overset{\circ}{\mathbf{u}}_{k+1}, \overset{\circ}{\mathbf{u}}_{k+2} \perp \operatorname{grad} \gamma, \qquad \varphi_{k+1} = \varphi_{k+1}(t, x_1, x_2, x_3) \end{gathered} \tag{2.13}$$

we obtain

$$\begin{gathered} \operatorname{grad} \gamma \, [\mathbf{M}(\overset{\circ}{u}_{k+1} + \varphi_{k+1} \operatorname{grad} \gamma) + \mathbf{L} \mathbf{u}_{k+1}] = 0 \\ \overset{\circ}{\mathbf{u}}_{k+2} = -\frac{\mathbf{M}(\mathbf{u}_{k+1}) + \mathbf{L}(\mathbf{u}_k)}{(\lambda + \mu)(\operatorname{grad} \gamma)^2} \end{gathered} \tag{2.14}$$

The first of these equations may be rewritten

$$\begin{gathered} 2\rho (\operatorname{grad} \gamma)^2 \left(\frac{\partial}{\partial t} \varphi_{k+1} \frac{\partial}{\partial t} \gamma - a^2 \operatorname{grad} \varphi_{k+1} \operatorname{grad} \gamma \right) + A \varphi_{k+1} + \\ + \operatorname{grad} \gamma (\mathbf{M}(\overset{\circ}{\mathbf{u}}_{k+1}) + \mathbf{L}(\mathbf{u}_{k+2})) = 0 \end{gathered} \tag{2.15}$$

where A is a regular function of the coordinates which does not depend on $\mathbf{u}_j$ and which will not be written explicitly, in order to save space.

If the equation $\mathbf{x} = \mathbf{x}(\tau_a)$ represents an extremal of the integral (2.10), then the curve in four-dimensional space which is given by the equations

$$\mathbf{x} = (x_1, x_2, x_3) = \mathbf{x}(\tau_a), \qquad t = \tau_a + \text{const}$$

will be a characteristic of the equation

$$\gamma_t^2 = a^2 (\operatorname{grad} \gamma)^2 \tag{2.16}$$

and, consequently, a bicharacteristic of Equation (1.1).

Equation (2.15) may be rewritten as follows:

$$2\rho\,(\operatorname{grad}\gamma)^2\frac{d\varphi_{k+1}}{d\sigma}+A\varphi_{k+1}+\operatorname{grad}\gamma\,(\mathbf{M}(\overset{\circ}{\mathbf{u}}_{k+1})+\mathbf{L}(\mathbf{u}_k))=0 \tag{2.17}$$

where

$$\frac{d}{d\sigma}=\frac{d}{dt}+\frac{d}{d\tau_a},\qquad \tau_a=\int\limits_{M_1}^{M}\frac{ds}{a}$$

The derivative $d/d\sigma$ is indeed a time derivative, taken along a bicharacteristic. Equations (2.13), (2.14) and (2.17) enable us to determine $\mathbf{u}_{k+1}$ and $\mathbf{u}^{\circ}_{k+2}$ once the initial conditions for Equation (2.17) are prescribed. Thus, all the vectors $\mathbf{u}_k$ of the sequence may be determined.

Consider now the case of the transverse wave. We shall seek the components of the unknown vectors along the directions of the vectors $\mathbf{x}_\tau$, x_α, x_β (see Equation (2.12)) as functions of the coordinates a, β, τ; the operator $\mathbf{M}$ of (2.5) takes the form

$$\begin{aligned}\mathbf{M}(\mathbf{u})=2\rho\frac{\partial\mathbf{u}}{\partial t}\frac{\partial\gamma}{\partial t}-(\lambda+\mu)\,(2\operatorname{div}\mathbf{u}\operatorname{grad}\gamma+\mathbf{u}\Delta\gamma)-\\-\frac{\mu}{b}\left[2\,|\operatorname{grad}\gamma\,|\frac{\partial\mathbf{u}}{\partial\tau}+\mathbf{u}b\,(\Delta\tau\gamma_\tau+\frac{1}{b^2}\gamma_{\tau\tau})\right]-\\-\operatorname{grad}\lambda\,(\mathbf{u}\operatorname{grad}\gamma)-(\operatorname{grad}\mu\mathbf{u})\operatorname{grad}\gamma-(\operatorname{grad}\mu\operatorname{grad}\gamma)\,\mathbf{u}\end{aligned} \tag{2.18}$$

From Equations (2.3) to (2.8) it follows that $\mathbf{u}_{-2}\equiv 0$, $\mathbf{u}_{-1}\equiv 0$, $\mathbf{u}_0^{\circ}\equiv 0$, where $\mathbf{u}_0^{\circ}$ denotes the component of the vector $\mathbf{u}_0$ along the ray.

Now suppose that the vectors $\mathbf{u}_{-2}$, $\mathbf{u}_{-1}$, $\mathbf{u}_0$, ..., $\mathbf{u}_k$, $\mathbf{u}^{\circ}_{k+1}$ are known (where $\mathbf{u}^{\circ}_{k+1}$ is the component of the vector $\mathbf{u}_{k+1}$ along the ray). Then

$$\begin{gathered}\mathbf{u}_{k+1}=\overset{\circ}{\mathbf{u}}_{k+1}+u_{k+1,\alpha}\boldsymbol{\alpha}+u_{k+1,\beta}\boldsymbol{\beta}\\ \overset{\circ}{\mathbf{u}}_{k+1}\,\|\operatorname{grad}\tau_b,\qquad \boldsymbol{\alpha}=\frac{\mathbf{x}_\alpha}{|\mathbf{x}_\alpha|},\qquad \boldsymbol{\beta}=\frac{\mathbf{x}_\beta}{|\mathbf{x}_\beta|}\end{gathered} \tag{2.19}$$

From Equations (2.3) and (2.8) it follows that

$$\mathbf{M}(\mathbf{u}_{k+1})+\mathbf{L}(\mathbf{u}_k)\,\|\operatorname{grad}\gamma$$

or, what is the same

$$\frac{2\mu}{b^2}\left(\frac{du_{k+1,\alpha}}{d\sigma}+(\boldsymbol{\alpha\beta})\frac{du_{k+1,\beta}}{d\sigma}\right)+Au_{k+1,\alpha}+Bu_{k+1,\beta}=[\mathbf{L}(\mathbf{u}_k)+\mathbf{M}(\overset{\circ}{\mathbf{u}}_{k+1})]\,\boldsymbol{\alpha}$$

$$\frac{2\mu}{b^2}\left(\frac{du_{k+1,\alpha}}{d\sigma}(\alpha\beta)+\frac{du_{k+1,\beta}}{d\sigma}\right)+Cu_{k+1,\alpha}+Du_{k+1,\beta}=[\mathbf{L}(\mathbf{u}_k)+\mathbf{M}(\overset{\circ}{\mathbf{u}}_{k+1})]\,\beta$$

$$\frac{d}{d\sigma}=\frac{d}{dt}+\frac{d}{d\tau_b},\quad \tau_b=\int_{M_1}^{M}\frac{ds}{b} \tag{2.20}$$

where A, B, C, D denote regular functions whose explicit expressions are omitted.

If (2.20) holds, then (2.3) implies that

$$\overset{\circ}{\mathbf{u}}_{k+2}=\frac{L(\mathbf{u}_k)+\mathbf{M}(\mathbf{u}_{k+1})}{(\lambda+\mu)(\operatorname{grad}\gamma)^2} \tag{2.21}$$

3. Construction of the solution of the Cauchy problem (1.7), (1.8). Let us now denote by $\gamma_{\omega a}$ ($\gamma_{\omega b}$) the solution of the equation

$$(\operatorname{grad}\gamma)^2=\frac{1}{a^2}\gamma_t^2\quad\left(=\frac{1}{b^2}\gamma_t^2\right) \tag{3.1}$$

satisfying the following conditions:

$$\gamma_a=(x_l-x_l^\circ)\,\omega_l \text{ for } t=0,\ \gamma_t>0;\qquad \gamma_b=(x_l-x_l^\circ)\,\omega_l \text{ for } t=0,\ \gamma_t>0$$

We shall seek the solution of the Cauchy problem (1.7) and (1.8) in the form

$$\mathbf{h}_\omega=\sum_{k=0}^{\infty}\mathbf{u}_{\omega ka}(x_1,x_2,x_3,t)\,[f_k(\gamma_a(t,x_1,x_2,x_3))-f_k(\gamma_a(-t,x_1,x_2,x_3))]+$$

$$+\sum_{k=0}^{\infty}\mathbf{u}_{\omega kb}(x_1,x_2,x_3,t),\ [f_k(\gamma_b(t,x_1,x_2,x_3))-f_k(\gamma_b(-t,x_1,x_2,x_3))] \tag{3.2}$$

where $\mathbf{u}_{\omega ka}$ is determined by means of the recurrence relations (2.13) to (2.17) and $\mathbf{u}_{\omega kb}$ is determined by means of Formulas (2.18) to (2.21).

The initial condition $\mathbf{h}_\omega = 0$ for $t = 0$ and Equation (1.1) with $\mathbf{K} = 0$ are obviously satisfied. Let us consider the second initial condition (1.8). We have

$$2\sum_{k=0}^{\infty}(\mathbf{u}_{\omega ka}a+\mathbf{u}_{\omega kb}b)\,f_{k-1}((x_l-\overset{\circ}{x_l})\,\omega_l)=-\frac{1}{8\pi^2\rho(M_0)}\delta''((x_l-x_l^\circ)\,\omega_l)\,i_j$$

$$\text{for } t=0 \tag{3.3}$$

In order that this equation be satisfied it is sufficient that the first term in the sum equal the right-hand side and that the remaining terms on the left-hand side equal zero.

It is natural to put $f_0(t) = \delta'(t)$. Then

$$f_1(t) = \delta(t), \qquad f_2(t) = \varepsilon(t), \ldots, f_k(t) = \frac{t_+^{k-2}}{(k-2)!} = \frac{t^{k-2}\varepsilon(t)}{(k-2)!}$$

where $\epsilon(t)$ is the Heaviside function, which equals one for $t \geqslant 0$ and zero for $t < 0$.

In order that (3.3) hold it is then sufficient to require that

$$2\mathbf{u}_{\omega 0a}a + 2\mathbf{u}_{\omega 0b}b = -\frac{\mathbf{i}_j}{8\pi^2\rho(M_0)} \qquad \text{for } t = 0 \tag{3.4}$$

$$\mathbf{u}_{\omega ka}a + \mathbf{u}_{\omega kb}b = 0 \qquad (k > 0) \qquad \text{for } t = 0 \tag{3.5}$$

For $t = 0$ the solutions γ_a and γ_b coincide. For $t = 0$ the vector $\mathbf{u}_{\omega 0a}$ is parallel to grad $\gamma_a = \omega$ and the vector $\mathbf{u}_{\omega 0b}$ is perpendicular to $\boldsymbol{\omega}$. By decomposing the vector on the right-hand side of Equation (3.4) into its components in the direction of $\boldsymbol{\omega}$ and perpendicular to $\boldsymbol{\omega}$ we obtain uniquely defined initial data for Equations (2.17) and (2.20) for $k = -1$, and then $\mathbf{u}_{\omega 0a}$ and $\mathbf{u}_{\omega 0b}$, and $\mathbf{u}^{\circ}_{\omega 0a}$ and $\mathbf{u}^{\circ}_{\omega 0b}$ are uniquely determined. Substituting the following two equations into (3.5)

$$\mathbf{u}_{\omega 1a} = (\varphi_{\omega 1}\,\mathrm{grad}\,\gamma + \overset{\circ}{\mathbf{u}}_{\omega 1a}), \quad \mathbf{u}_{\omega 1b} = (u_{\omega 1a}\boldsymbol{\alpha} + \mathbf{u}_{\omega 1\beta}\boldsymbol{\beta} + \overset{\circ}{\mathbf{u}}_{\omega 1b})$$

we obtain unique initial data for $\phi_{\omega 1}$, $u_{\omega 1a}$ and $\mathbf{u}_{\omega 1\beta}$; at the same time $\mathbf{u}_{\omega 1a}$, $\mathbf{u}_{\omega 1b}$, $\mathbf{u}^{\circ}_{\omega 2a}$, $\mathbf{u}^{\circ}_{\omega 2b}$ are uniquely determined, etc.

Thus, all the vectors $\mathbf{u}_{\omega ka}$ and $\mathbf{u}_{\omega kb}$ are uniquely determined and are analytic functions. The convergence of the series (3.2) can be established, using the method of majorants, exactly as in the case of the general hyperbolic equation with non-multiple characteristics.

It is easy to show that $\gamma(-t, x_1, x_2, x_3, \omega) = -\gamma(t, x_1, x_2, x_3, \omega)$.

Using these relations we obtain finally

$$\mathbf{h}_j = \int\limits_{|\omega|=1} \mathbf{h}_{j\omega}d\omega = \int\limits_{|\omega|=1} [\mathbf{u}_{\omega 0aj}(t, x_1, x_2, x_3)\,\delta'(\gamma_{\omega a}) +$$

$$+ \mathbf{v}_{\omega aj}(t, x_1, x_2, x_3)\,\varepsilon(\gamma_{\omega a})]\,d\omega + \int\limits_{|\omega|=1} [\mathbf{u}_{\omega 0bj}(t, x_1, x_2, x_3)\,\delta'(\gamma_{\omega b}) +$$

$$+ \mathbf{v}_{\omega bj}(t, x_1, x_2, x_3)\,\varepsilon(\gamma_{\omega b})]\,d\omega = \mathbf{h}_{aj} + \mathbf{h}_{bj} \tag{3.6}$$

$$\varepsilon(\gamma)=\begin{cases}1 & \text{for } \gamma\geqslant 0\\ 0 & \text{for } \gamma<0\end{cases}$$

where $\mathbf{v}_{\omega aj}$ and $\mathbf{v}_{\omega bj}$ are vectors with regular components.

4. On the singularities of the fundamental tensor. Thus, the fundamental tensor is expressed, by means of (3.6), as a sum of generalized plane waves of the form (3.2).

Using Equation (3.6) the analytic properties of the fundamental tensor may be studied. We shall content ourselves here with a statement of the results, inasmuch as, using the methods of Borovikov [9], the author studied the fundamental tensor of an arbitrary system which is hyperbolic in Petrovskii's sense [2,11] and the analysis of the fundamental tensor in our present case may be carried out analogously.

In particular, the components of the fundamental tensor

$$H(M,\ M_0,\ t)=\|h_{ij}\|;\qquad M=M(x_1,\ x_2,\ x_3),\qquad M_0=M_0(x_1^\circ,\ x_2^\circ,\ x_3^\circ)$$

are defined in the neighborhood of the point M_0 and are equal to zero for $t<\tau_a(M,\ M_0)$.

For $\tau_a<t<\tau_b$ and $\tau_b<t$ they are analytic functions of their arguments, and for $t=\tau_a$ and $t=\tau_b$ they have a δ-function-like singularity, namely

$$\begin{aligned}h_{jk}(t,\ M,\ M_0)&=V_{jka}\delta(t-\tau_a)+V_{jkb}\delta(t-\tau_b)+\\&+W_{jka}\varepsilon(t-\tau_a)+W_{jkb}\varepsilon(t-\tau_b)\qquad (j,\ k=1,\ 2,\ 3)\end{aligned}\tag{4.1}$$

where V_{jka}, V_{jkb}, W_{jka}, W_{jkb} are regular functions of t, M, M_0, and ϵ is Heaviside's function.

In the plane case the equations of elasticity are hyperbolic in Petrovskii's sense, since the non-multiple characteristics are then absent. The fundamental tensor for systems which are hyperbolic in Petrovskii's sense was obtained in [2] and [11]. By means of considerations similar to those just carried out in the three-dimensional case, one arrives, instead of (4.1), at a formula where the singularity of (4.1) is replaced by a singularity of the form $1/\sqrt{x}$, namely

$$h_{jk}=V_{jka}(t,\ M,\ M_0)(t-\tau_a)_+^{-1/2}+V_{jkb}(t,\ M,\ M_0)(t-\tau_b)_+^{-1/2}$$

$$x_+^{-1/2}=\begin{cases}x^{-1/2} & (x>0)\\ 0 & (x\leqslant 0)\end{cases}\tag{4.2}$$

where V_{jka} and V_{jkb} are regular functions of their arguments.

If the Lamé parameters λ and μ and the density ρ are not analytic functions but are just sufficiently smooth functions of the coordinates, then the considerations of the present section remain in force, while the functions V and W are than no longer analytic but merely sufficiently smooth functions of their arguments.

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